Conjugate Degradability and the Quantum Capacity of Cloning Channels

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Abstract

A quantum channel is conjugate degradable if the channel's environment can be simulated up to complex conjugation using the channel's output. For all such channels, the quantum capacity can be evaluated using a single-letter formula. In this article we introduce conjugate degradability and establish a number of its basic properties. We then use it to calculate the quantum capacity of N to $N+1$ and 1 to M universal quantum cloning machines as well as the quantum capacity of a channel that arises naturally when data is being transmitted to an accelerating receiver. All the channels considered turn out to have strictly positive quantum capacity, meaning they could be used as part of a communication system to send quantum states reliably.

I. INTRODUCTION

The quantum capacity of a noisy quantum channel measures the channel's ability to transmit quantum coherence. The capacity is defined by optimizing over all possible quantum error correcting codes in the limit of many uses of the channel and with the provision that decoding errors ultimately become negligible. Because of the ubiquity of noise in quantum systems, error correction will necessarily play a central role in future attempts to build devices exploiting quantum information, which makes finding a general expression for the quantum capacity, and the codes associated with it, one of the most important problems in quantum information theory.

To date, the best general lower bound on the capacity is given by optimizing a quantity known as the (one-shot) coherent information [10, 20, 22, 23]. Unfortunately, it is also known that for some channels this prescription fails to yield capacity [12, 24]. For at least one class of channels, however, known as the *degradable* channels, the prescription *does* yield the quantum capacity and, therefore, the optimal codes [11]. Degradable channels were introduced by Devetak and Shor, building on ideas from classical information theory [2]. Roughly speaking, they are channels for which the output of the channel always contains at least as much information as the channel's environment, in the sense that the channel environment can be simulated using the channel output. Many familiar channels are in fact degradable, such as generalized dephasing channels, erasure channels with erasure probability at most 1/2 and Kraus diagonal channels [9]. If the role of the channel output and the environment are reversed in the definition, the channel is called *antidegradable*. By a simple no-cloning argument, such channels always have zero quantum capacity. A host of

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familiar channels are in this class, including all entanglement-breaking channels. Indeed, nearly every channel for which the quantum capacity is known is either degradable or antidegradable.

In this article, we expand the list of quantum channels for which the quantum capacity can be calculated by introducing the property of *conjugate degradability*. Conjugate degradability is defined much like degradability except that the channel output can be used to simulate the environment only up to complex conjugation or, equivalently, transposition. Many desirable properties of degradability continue to hold despite this modification, including, for example, the fact that the optimized coherent information is equal to the quantum capacity.

Conjugate degradability first appeared implicitly in [4], which analyzed private data transmission in the presence of a uniformly accelerated eavesdropper. When the data is encoded using a bosonic, dual-rail qubit, the effective channel to the eavesdropper is a conjugate degradable channel with infinite-dimensional output. Not all examples are so exotic, however. Optimal $N \to N+1$ and $1 \to M$ cloning machines are also conjugate degradable, as we will demonstrate before using the property to evaluate their capacities. Understanding coherence preservation in these channels is particularly interesting given the well-known connection between detecting separability and the existence of symmetric extensions [13]; cloning channels attempt to construct symmetric extensions for arbitrary states. We find a strictly positive general formula for the quantum capacities of $N \to N+1$ and $1 \to M$ universal quantum cloning machine (UQCM), establishing that all such machines are useful for transmitting quantum information reliably.

Structure of the paper: Section II introduces the notion of conjugate degradability and demonstrates that there is a single-letter formula for the quantum capacity of conjugate degradable channels. Section III uses conjugate degradability to evaluate the quantum capacity of Unruh channels as well as the universal quantum cloning machines. Next, section IV establishes a number of structural properties of conjugate degradable channels following the template for degradable channels established by Cubitt *et al.* [9]. Finally, in an appendix, we present some supporting evidence in favor of the conjecture that all $N \to M$ universal quantum cloning machines are conjugate degradable.

II. THE QUANTUM CAPACITY OF CONJUGATE DEGRADABLE CHANNELS

A. Notation

Before we proceed, let us recall some relevant concepts from quantum information theory and fix some notation. Let $\mathcal{B}(\mathcal{H})$ represent the space of bounded linear operators on the Hilbert space $\mathcal{H}$. A quantum channel is a completely positive, trace-preserving (CPTP) map between two such spaces of operators. We will typically call the input space A and the output space B so that the quantum channel is a map $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$. The identity channel will be denoted by $\mathbb{I}$.

Every quantum channel has a *Stinespring dilation*, a representation of the channel in which the action of the channel on an input density matrix ϱ is given by $\mathcal{N}(\varrho) = \mathrm{Tr}_E(U\varrho U^\dagger)$, where $U : A \to B \otimes E$ satisfies $U^\dagger U = I$. E labels an auxiliary space usually called the environment because it models the effect of noise on the channel. There is also a complementary channel $\mathcal{N}^c : \mathcal{B}(A) \to \mathcal{B}(E)$, given by tracing over the output space instead of the environment: $\mathcal{N}^c(\varrho) = \mathrm{Tr}_B(U\varrho U^\dagger)$.

Throughout, we will use subscripts to label subsystems, so that $\varrho_A = \mathrm{Tr}_B\, \varrho_{AB}$, for example. The density operator of a pure state $|\psi\rangle$ will sometimes be denoted by ψ. $H(\tau) = -\mathrm{Tr}(\tau \log \tau)$ is the von Neumann entropy of a density matrix τ. Sometimes it is more convenient (or important) to emphasize the particular system on which the density matrices act. In that case, we will write $H(X)_\tau = H(\tau_X)$ to denote the von Neumann entropy of the state τ restricted to the space X. The

conditional entropy $H(X|Y)_\tau$ is defined to be $H(XY)_\tau - H(Y)_\tau$.

The *coherent information* of a quantum channel for a given input density matrix is given by $I_c(\mathcal{N}, \varrho) = H(\mathcal{N}(\varrho)) - H(\mathcal{N}^c(\varrho))$. Equivalently, given the Stinespring dilation U for $\mathcal{N}$ and setting $\tau = U\varrho U^\dagger$, $I_c(\mathcal{N}, \varrho) = H(B)_\tau - H(E)_\tau$. As is evident from the formula, the coherent information measures the reduction via leakage into the environment of the information transmitted about ϱ . The quantum capacity of a channel $\mathcal{N}$ is given by the limit

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}), \tag{1}$$

where $Q^{(1)}(\mathcal{N})$ is the maximum output coherent information of a channel, given by

$$Q^{(1)}(\mathcal{N}) = \max_{\varrho} I_c(\mathcal{N}, \varrho).$$

B. Degradable quantum channels

We can now introduce the *degradability* of quantum channels. A channel $\mathcal{N}$ is called *degradable* if there exists a CPTP map, $\mathcal{D} : \mathcal{B}(B) \to \mathcal{B}(E)$ such that $\mathcal{D} \circ \mathcal{N} = \mathcal{N}^c$. Similarly, a channel is *antidegradable* if its complementary channel is degradable, or equivalently, if there exists a CPTP map $\mathcal{D}^a$ such that $\mathcal{D}^a \circ \mathcal{N}^c = \mathcal{N}$. This has been shown in the accompanying diagram: As mentioned

$$\begin{tikzcd} A \arrow[r, "\mathcal{N}"] \arrow[dr, "\mathcal{N}^c"'] & B \arrow[d, dotted, shift right, "\mathcal{D}"'] \\ & E \arrow[u, dotted, shift right, "\mathcal{D}^a"'] \end{tikzcd}$$

above, degradable channels are interesting because their quantum capacities are easily evaluated. The reason for this ease is the observation in [11] that for degradable channels, the optimized coherent information is additive. In other words, $Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N})$. This means that the quantum capacity in (1) can be computed using the *single-letter* formula $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = \max_\varrho I_c(\mathcal{N}, \varrho)$, as compared to the general case in which essentially nothing is known about the rate of convergence to the limit.

C. Conjugate degradability

In [4], the authors investigated a channel with infinite dimensional output that was degradable up to complex conjugation of the output space. In other words, the channel wasn't obviously degradable but it nearly was. In this paper we generalize that example to the class of conjugate degradable channels, showing not only that there exist some important finite dimensional examples in this class but that the conjugate degradable channels share many of the useful properties of degradable channels with respect to their structure and capacities.

To understand conjugate degradability, it is helpful to consider the following diagram:

$$\begin{tikzcd}
& B \arrow[d, dotted, "\mathcal{D}"] \arrow[dr, "\mathcal{C}"] & \\
A \arrow[ur, "\mathcal{N}"] \arrow[dr, "\mathcal{N}^c"'] & E' & B' \\
& E \arrow[u, "\mathcal{C}"'] \arrow[ur, dotted, "\mathcal{D}^a"'] &
\end{tikzcd}$$

As above, A, B, E play the role of the input, output and the environment. Let $\mathcal{C} : \mathcal{B}(E) \to \mathcal{B}(E')$ denote entry-wise complex conjugation with respect to a fixed basis of $E \cong E'$. A quantum channel is called *conjugate degradable* if there exists a CPTP map $\mathcal{D} : \mathcal{B}(B) \to \mathcal{B}(E')$ such that the diagram commutes. In other words,

$$\mathcal{D} \circ \mathcal{N} = \mathcal{C} \circ \mathcal{N}^c. \tag{2}$$

In a similar fashion, $\mathcal{N}$ is *conjugate antidegradability* if there exists a CPTP conjugate degrading map $\mathcal{D}^a$ for the complementary channel giving

$$\mathcal{D}^a \circ \mathcal{N}^c = \mathcal{C} \circ \mathcal{N}. \tag{3}$$

It is immediate from the definition that $\mathcal{N}$ is conjugate degradable if and only if $\mathcal{N}^c$ is conjugate antidegradable.

Let us now consider what happens during the computation of the quantum capacity for conjugate degradable channels. The map $\mathcal{D} : \mathcal{B}(B) \to \mathcal{B}(E')$ can also be represented by its Stinespring dilation with the addition of another auxiliary space, say F, so that we have a corresponding isometry $V : B \to E'F$ in addition to the isometry $U : A \to BE$ for $\mathcal{N}$. Thus $H(B)_\tau = H(E'F)_\omega$ for the states $\tau = U\varrho U^\dagger$ and $\omega = V\tau V^\dagger$. Now consider the coherent information of the channel, $I_c(\mathcal{N}, \varrho) = H(B)_\tau - H(E)_\tau = H(E'F)_\omega - H(E')_\omega = H(F|E')_\omega$, using the fact that complex conjugation does not change the entropy. For two uses of the channel $\mathcal{N}^{\otimes 2} : \mathcal{B}(A_1A_2) \to \mathcal{B}(B_1B_2)$, repeated application of the strong subadditivity of von Neumann entropy yields

$$\begin{aligned} H(F_1F_2|E_1'E_2')_\omega &\leq H(F_1|E_1'E_2')_\omega + H(F_2|E_1'E_2')_\omega \\ &\leq H(F_1|E_1')_\omega + H(F_2|E_2')_\omega \end{aligned}$$

where $\omega = (V \otimes V)\tau(V^\dagger \otimes V^\dagger)$ and $\varrho = (U \otimes U)\varrho_{A_1A_2}(U^\dagger \otimes U^\dagger)$ for any $\varrho_{A_1A_2}$. This shows that $I_c(\mathcal{N}^{\otimes 2}, \varrho_{A_1A_2}) \leq I_c(\mathcal{N}, \varrho_{A_1}) + I_c(\mathcal{N}, \varrho_{A_2})$, which in turn implies that $Q^{(1)}(\mathcal{N}^{\otimes 2}) \leq 2Q^{(1)}(\mathcal{N})$. Since the superadditivity is an immediate consequence of the definition and we can easily deduce by induction that $Q^{(1)}(\mathcal{N}^{\otimes n}) \leq nQ^{(1)}(\mathcal{N})$, we clearly have, just as in the case of degradable channels, a single-letter expression for the capacity of a conjugate degradable channel,

$$Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = \max_\varrho I_c(\mathcal{N}, \varrho). \tag{4}$$

Next, consider the class of conjugate antidegradable channels. Suppose that it is possible to send quantum states through such a channel in the sense that there is a decoding CPTP map $\mathcal{R}$ such that $(\mathcal{R} \circ \mathcal{N})(\phi)$ has high fidelity with $|\phi\rangle$ for all $|\phi\rangle$ in a subspace of dimension at least 2. By the definition of conjugate antidegradability, the output $\mathcal{N}^c(|\phi\rangle\langle\phi|)$ of the complementary channel

can be transformed to a high-fidelity copy of the conjugated state $|\bar{\phi}\rangle$ using the antidegrading map $\mathcal{D}^a$. Combining these two operations, from A to B and from A to the conjugated B' via the complementary channel, approximately implements the map $|\phi\rangle \mapsto |\phi\rangle\,|\bar{\phi}\rangle$. But this transformation is nonlinear and is therefore not possible. It follows that the transmission of the quantum states with high fidelity over the antidegradable quantum channel is not possible. Likewise, the quantum capacity of all conjugate antidegradable channels is identically zero.

Referring back to the previous diagram, if $\mathcal{N}$ is conjugate degradable, then for any state ϱ_{AR},

$$(\mathcal{C} \otimes \mathrm{id})(\mathcal{N}^{\mathrm{c}} \otimes \mathrm{id})\varrho_{\mathrm{AR}} = (\mathcal{D} \circ \mathcal{N} \otimes \mathrm{id})\varrho_{\mathrm{AR}} \geq 0.$$

Thus, the state $\sigma_{ER} = (\mathcal{N}^c \otimes \mathrm{id})\varrho_{\mathrm{AR}}$ has a positive partial transpose. (Complex conjugation and transpose act identically on Hermitian matrices.) From [17], there are only two possibilities:

1. σ_{ER} is a convex combination of product states (also known as a separable state), which is equivalent to saying that $\mathcal{N}^c$ is entanglement-breaking. Since entanglement-breaking channels are a subclass of antidegradable channels, this implies that $\mathcal{N}$ is degradable.

2. σ_{ER} is a bound entangled state, which is equivalent to saying that $\mathcal{N}^c$ is entanglement-binding.

Moreover, bound entanglement does not exist in quantum systems of total dimension less than or equal to six [17]. As a result, conjugate antidegradable channels with input and output dimensions satisfying this constraint are also entanglement-breaking. As a special case, suppose $\mathcal{N}$ is a conjugate degradable qubit channel (input and output dimension equal to two). We will prove in Theorem 4 that the Choi rank of conjugate degradable qubit channels is less than or equal to two. This means that $\mathcal{N}^c$ is also a qubit channel, which implies that the product of $\mathcal{N}^c$'s input and output space dimensions is less than six. But since $\mathcal{N}^c$ is also conjugate antidegradable, $\mathcal{N}^c$ has to be entanglement-breaking.

III. EXAMPLES

A. Example 1: $1 \to 2$ universal quantum cloning machine

While cloning quantum states is impossible, approximate cloning is permitted provided the quality of the clones is sufficiently poor. A $1 \to 2$ universal quantum cloner is a CPTP map $\mathcal{C}l : \mathcal{B}(A) \to \mathcal{B}(B_1 B_2)$, where A, B_1 and B_2 are all qubits such that

- $\mathrm{Tr}_{B_1} \mathcal{C}l = \mathrm{Tr}_{B_2} \mathcal{C}l$: the outputs on B_1 and B_2 are identical.

- The Uhlmann fidelity [18, 25] $F(\psi, \mathrm{Tr}_{B_1} \mathcal{C}l(\psi))$ is independent of the input state ψ.

The optimal cloning machine maximizes the fidelity $F(\psi, \mathrm{Tr}_{B_1} \mathcal{C}l(\psi))$. Bužek and Hillery [7] constructed such a machine (shown to be optimal in [5]), which we will see provides a natural example of a conjugate degradable channel. Later we will study the more complicated case of $N \to M$ cloning.

Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. The Stinespring dilation of $\mathcal{C}l$ acts as follows:

$$U : |\psi\rangle_A \mapsto \sqrt{\frac{2}{3}}\,|\psi\rangle_{B_1}|\psi\rangle_{B_2}|\bar{\psi}\rangle_E + \sqrt{\frac{1}{6}}\,|\psi^\perp\rangle_{B_1}|\psi\rangle_{B_2}|\bar{\psi}^\perp\rangle_E + \sqrt{\frac{1}{6}}\,|\psi\rangle_{B_1}|\psi^\perp\rangle_{B_2}|\bar{\psi}^\perp\rangle_E\,, \quad (5)$$

where $|\bar{\psi}\rangle = \bar{\alpha}\,|0\rangle + \bar{\beta}\,|1\rangle$, $|\psi^{\perp}\rangle = -\bar{\beta}\,|0\rangle + \bar{\alpha}\,|1\rangle$ and $|\bar{\psi}^{\perp}\rangle = -i\sigma_Y\,|\psi\rangle = -\beta\,|0\rangle + \alpha\,|1\rangle$. Taking the partial trace over either B_2E or B_1E yields the density operator of the individual clones:

$$\varrho_{B_j} = \frac{5}{6}|\psi\rangle\langle\psi| + \frac{1}{6}|\psi^{\perp}\rangle\langle\psi^{\perp}|. \tag{6}$$

The output of the complementary channel to the environment is given by

$$\varrho_E = \frac{1}{3}|\bar{\psi}^{\perp}\rangle\langle\bar{\psi}^{\perp}| + \frac{2}{3}|\bar{\psi}\rangle\langle\bar{\psi}|. \tag{7}$$

Another way to visualize this transformation is that the input density matrix $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{I} + \hat{n}\cdot\vec{\sigma})$ gets mapped to

$$\varrho_{B_j} = \frac{1}{2}\Big(\mathbb{I} + \frac{2}{3}\hat{n}\cdot\vec{\sigma}\Big), \quad \varrho_E = \frac{1}{2}\Big(\mathbb{I} - \frac{1}{3}\big(-n_x\sigma_x + n_y\sigma_y - n_z\sigma_z\big)\Big), \tag{8}$$

where $\hat{n} \in \mathbb{R}^3$ is the unit Bloch vector and $\vec{\sigma}$ is a vector of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. From this, it is straightforward to build a conjugate degrading map from the output B_1 to the environment E by first depolarizing to shrink the Bloch vector by a factor of 2, followed by complex conjugation. Although this does not imply degradability because of the needed antiunitary operation of conjugation, it does prove that this channel is conjugate degradable. (Below we will show that $\mathcal{C}l$ is actually also degradable.)

We now compute the capacity of this channel. Setting $\tau = U\psi U^{\dagger}$, Eq. (4) states that the quantum capacity is equal to $\max_{\psi}[H(B_1B_2)_{\tau} - H(E)_{\tau}]$. Again writing $V : B \to E'F$ for the Stinespring dilation of the conjugate degrading map $\mathcal{D}$ and $\omega = V\tau V^{\dagger}$, we also saw that the capacity could be rewritten as

$$\max_{\psi}[H(FE')_{\omega} - H(E')_{\omega}] = \max_{\psi} H(F|E')_{\omega}.$$

For a fixed input ψ, let $\omega(\psi) = VU\psi U^{\dagger}V^{\dagger}$ denote the dependence of ω on this input. Due to the unitary covariance of the channel and the concavity of the conditional entropy,

$$H(F|E')_{\omega(\psi)} = \int_{\mathcal{U}(A)} H(F|E')_{\omega(W\psi W^{\dagger})}dW \le H(F|E')_{\omega(\int W\psi W^{\dagger}dW)} = H(F|E')_{\omega(\mathbb{I}/2)}, \tag{9}$$

where $\mathbb{I}/2$ is the maximally mixed state at the input. Thus the coherent information is maximized when the input is maximally mixed. Substituting into the formula reveals the quantum capacity of the $1 \to 2$ cloner to be

$$Q(\mathcal{C}l) = H(B_1B_2)_{\Pi^+/3} - H(E)_{\mathbb{I}/2} = \log 3 - 1, \tag{10}$$

where Π^+ is the projector onto the symmetric subspace of B_1B_2.

As indicated earlier, $\mathcal{C}l$ is actually also degradable, which gives an alternative method for justifying the use of the single-letter capacity formula. Demonstrating degradability is somewhat more delicate than conjugate degradability, however. For the purposes of illustration, we sketch the argument here. The universal not ($UNOT$) operation is the antiunitary transformation that negates a qubit's Bloch vector. From Eq. (8), the necessary degrading map will have the form $\mathcal{D} = \mathrm{Tr}_{B_2} \circ UNOT \circ \mathcal{S} \circ \mathcal{P}^+$ where $\mathcal{S}$ is a depolarizing map shrinking the Bloch vector by a factor of 2 and $\mathcal{P}^+(\varrho) = \Pi^+\varrho\Pi^+$. ($\mathcal{D}$ is uniquely determined to be $(\mathcal{C}l^c) \circ \mathcal{C}l^{-1}$ on the range of $\mathcal{C}l$, to which the map is effectively restricted by the inclusion of $\mathcal{P}^+$.) The Jamiołkowski representation [8, 19] can then be used to test the complete positivity of $\mathcal{D}$. Letting $|\Phi\rangle = \frac{1}{2}(|00\rangle_{B_1B_1'} + |11\rangle_{B_1B_1'}) \otimes (|00\rangle_{B_2B_2'} + |11\rangle_{B_2B_2'})$, a tedious but straightforward calculation reveals that $(\mathcal{D} \otimes \mathrm{id}_{B_1'B_2'})(\Phi)$ is positive semidefinite. In fact, the map is only just positive semidefinite: if the Bloch vector had been shrunk by a factor less than 2 then $\mathcal{D}$ would not be completely positive.

B. Example 2: Unruh channel

Quantum field theory predicts that the vacuum state as defined by an inertial observer will be a thermal state from the point of view of a uniformly accelerated observer, an observation known as the Unruh effect [26]. Ref. [4] examined the transformation induced on a dual rail bosonic qubit channel when the receiver is uniformly accelerated, finding that the transformation can be represented by a channel

$$\mathcal{N}(|\psi\rangle\langle\psi|) = (1-z)^3 \bigoplus_{k=0}^{\infty} z^k \varrho_k \tag{11}$$

where $z \in (0,1)$, with increasing z corresponding to increasing acceleration. If $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{I} + \hat{n}\cdot\vec{\sigma})$ then

$$\varrho_k = \frac{(k+1)\mathbb{I}^{(k+2)}}{2} + \hat{n}\cdot\vec{J}^{(k+2)} \tag{12}$$

where $\vec{J}^{(k+2)} = (J_x^{(k+2)}, J_y^{(k+2)}, J_z^{(k+2)})$ are generators of the $(k+2)$-dimensional representation of $SU(2)$.

The channel was shown to be conjugate degradable and unitarily covariant in [4]. Therefore, just as for the $1 \to 2$ UQCM, the quantum capacity will be equal to $I_c(\mathcal{N}, \mathbb{I}/2)$. Setting $\tau_B = \mathcal{N}(\mathbb{I}/2)$ and $\tau_E = \mathcal{N}^c(\mathbb{I}/2)$ we find that

$$\tau_B = \bigoplus_{k=0}^{\infty} T_k S_k \mathbb{I}^{(k+2)} \quad \text{and} \quad \tau_E = \bigoplus_{k=0}^{\infty} T_k \tilde{S}_k \mathbb{I}^{(k+1)}, \tag{13}$$

where $T_k = (1-z)^3 z^k$, $S_k = \frac{1}{2}(k+1)$ and $\tilde{S}_k = \frac{1}{2}(k+2)$. The diagonal form of Eq. (13) is suitable for evaluating the von Neumann entropies in $I_c(\mathcal{N}, \mathbb{I}/2)$:

$$\begin{aligned}
Q(\mathcal{N}) &= H(B)_\tau - H(E)_\tau \\
&= -\sum_{k=0}^{\infty}\sum_{l=0}^{k+1} T_k S_k (\log T_k + \log S_k) + \sum_{k=0}^{\infty}\sum_{l=0}^{k} T_k \tilde{S}_k (\log T_k + \log \tilde{S}_k) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} T_k (k+1)(k+2)(\log \tilde{S}_k - \log S_k) \\
&= \frac{(1-z)^3}{2}\sum_{k=0}^{\infty} z^k (k+1)(k+2)(\log(k+2) - \log(k+1)) \\
&= \frac{(1-z)^3}{2}\sum_{k=0}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[z^{k+2}\log(k+2) - z z^{k+1}\log(k+1)\right] \\
&= \frac{(1-z)^3}{2}\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial s}\left[(z-1)\mathrm{Li}\,(s,z)\right]_{s=0},
\end{aligned}$$

where $\mathrm{Li}\,(s,z)$ is the polylogarithm function. The final expression gives a compact formula for the quantum capacity as a function of z or, equivalently, acceleration. Figure III B plots this function.

C. Example 3: $N \to M$ universal quantum cloning machines

An $N \to M$ universal quantum cloning machine $\mathcal{C}l_{N,M}$ is a generalization of the $1 \to 2$ UQCM $\mathcal{C}l$ described earlier. The input system A consists of N identical qubits. That is, A is the subspace

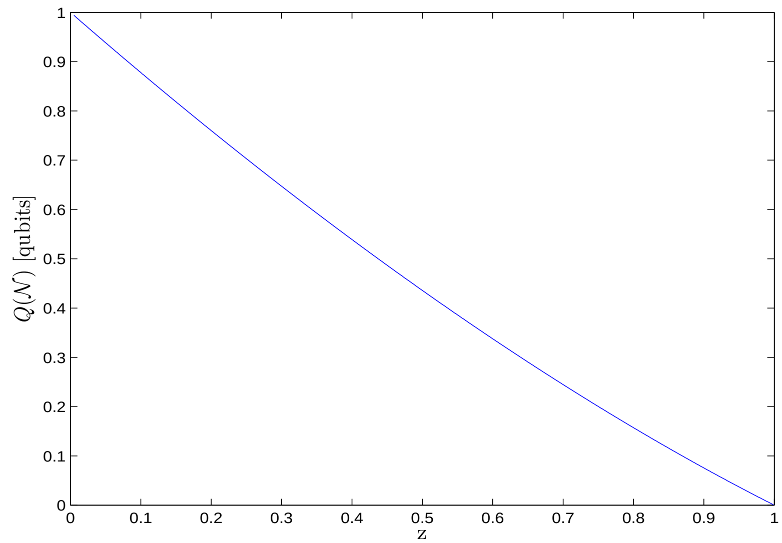

FIG. 1: Quantum capacity of the Unruh channel. Acceleration is monotonically increasing with z. The quantum capacity decreases from 1 for no acceleration to 0 for infinite acceleration, exhibiting a slight convexity as a function of z.

of an N-qubit system $A_1 A_2 \cdots A_N$ invariant under the natural action of the symmetric group S_N. The output system is the M-qubit space $B = B_1 B_2 \cdots B_M$, where $M \geq N$, and the cloner satisfies:

- $\mathrm{Tr}_{B_j} \mathcal{C}l = \mathrm{Tr}_{B_1} \mathcal{C}l$ for all j.
- The Uhlmann fidelity [18, 25] $F(\psi, \mathrm{Tr}_{B_1} \mathcal{C}l(\psi))$ is independent of the input state ψ.

Gisin and Massar constructed optimal $N \to M$ cloning machines [15] with the full optimality proof appearing in [6].

Theorem 1. *All $N \to N+1$ and $1 \to M$ universal quantum cloning machines for qubits are conjugate degradable.*

Proof. (i) $N \to N+1$ case. Tracing over any N of $N+1$ output qubits we find the following density matrix [28]

$$\varrho_{B_i} = \eta_{N,N+1} |\psi\rangle\langle\psi| + \frac{(1-\eta_{N,N+1})}{2} \mathbb{I}, \tag{14}$$

where $\eta_{N,N+1} = \frac{N}{N+1}\frac{N+3}{N+2}$. On the other hand, the environment is a single qubit system and of the following form

$$\varrho_E = \upsilon_N |\bar{\psi}\rangle\langle\bar{\psi}| + \frac{(1-\upsilon_N)}{2} \mathbb{I}, \tag{15}$$

where $\upsilon_N = N/(N+2)$. If we undo the complex conjugation and compare υ_N with the channel output we see that for all N (fixed) $\eta_{N,N+1} > \upsilon_N$ holds. This means that there always exists a CP map transforming ϱ_{B_i} into $\mathcal{C}(\varrho_E) \equiv \varrho_{E'}$ and thus the optimal $N \to N+1$ cloning machine is conjugate degradable.

(ii) $1 \to M$ *case.* We know that every UQCM is by definition a unitarily covariant map. If, for a given input state, we find a conjugate degrading map which is also unitarily covariant, we have a map working for all input states and the existence of a conjugate degrading has been proven. Let us fix the input qubit as $|\psi\rangle = |0\rangle$. Write $|N-j:0,j:1\rangle$ for a normalized completely symmetric state (CSS) of N qubits with exactly $N-j$ of them occupying state $|0\rangle$. Letting j range from 0 to N yields an orthonormal basis for the completely symmetric subspace, which has dimension $N+1$. The action of the Stinespring dilation of $\mathcal{C}l_{1,M}$ reads [15]

$$|1:0\rangle_A \xrightarrow{U_{1,M}} \sum_{j=0}^{M-1} \sqrt{\alpha_j(1,M)}\, |M-j:0,j:1\rangle_B\, |M-j-1:0,j:1\rangle_E$$

$$\equiv \sum_{j=0}^{M-1} \sqrt{\alpha_j(1,M)}\, |s_j\rangle_B\, |r_j\rangle_E\,, \quad (16)$$

where $\sqrt{\alpha_j(1,M)} = \frac{M-j}{\triangle_M}$ (see Eq. (47) in the appendix) and $\triangle_M = M(M+1)/2$ is a triangle number. Therefore,

$$\varrho_B = \frac{1}{\triangle_M}\bigoplus_{j=0}^{M-1}(M-j)|s_j\rangle\langle s_j| \quad \text{and} \quad \varrho_E = \frac{1}{\triangle_M}\bigoplus_{j=0}^{M-1}(M-j)|r_j\rangle\langle r_j|. \quad (17)$$

Since consecutive eigenvalues of these matrices differ by a constant increment, we can write both of them as linear combinations of the identity and a diagonal generator of the $su(2)$ algebra for the appropriate dimension. First, we will need this decomposition for the E subsystem so we write

$$\varrho_E = \frac{\mathbb{I}^{(M)}}{\triangle_M} + a_M J_z^{(M)}. \quad (18)$$

A swift calculation gives us $a_M = 1/\triangle_M$. Next, we introduce a new system $\tilde{B}$ which is given by tracing over any qubit of the B system. The state is supported by a CSS basis so by this procedure we get again a state supported by a lower-dimensional CSS basis. Moreover, the diagonality of the state from the B system is preserved. Hence, following the general case given by Eq. (49) from the appendix we get

$$\varrho_{\tilde{B}} = \frac{1}{\triangle_M}\bigoplus_{j=0}^{M-1}\frac{M^2+M-1-j(2+M)}{M}|r_j\rangle\langle r_j|. \quad (19)$$

Note that $\varrho_{\tilde{B}}$ lives in the same Hilbert space as ϱ_E and the diagonal coefficients of $\varrho_{\tilde{B}}$ are given by Eq. (50) from the appendix.

It is intuitively clear that when going from B to $\tilde{B}$ we transformed a J_z generator into another (lower-dimensional) J_z generator. If the intuition is not convincing we can see that the difference between two consecutive eigenvalues of Eq. (19) is equal to $\frac{1}{\triangle_M}\frac{2+M}{M}$ and indeed independent on j. We may therefore again write

$$\varrho_{\tilde{B}} = \frac{\mathbb{I}^{(M)}}{\triangle_M} + b_M J_z^{(M)}. \quad (20)$$

From (19) and (20) we find $b_M = \frac{1}{\triangle_M}\frac{M^2+M-2}{M(M-1)}$ and hence $a_M/b_M < 1, \forall M > 1$. Now what if the input state in Eq. (16) is a general qubit? The output space are not spanned by $|r_j\rangle\,, |s_j\rangle$ anymore. The complete output state of the transformation is explicitly dependent on the input state as can

be seen from Eq. (46) for $N=1$. Now we have to pay a special attention to the E subsystem. First of all, we introduce a complex conjugated system E' which is spanned by $\{|R_j\rangle\}_{j=0}^{M-N}$. Recall that the transformation $\varrho_E \mapsto \varrho_{E'}$ is not a unitary operation. But the subsystem E' is exactly the system where the output of a conjugate degrading map should appear (cf. Sec II C) and it is also the output space of the depolarizing map defined above due to its unitary covariance. We may conclude that we found an input-independent conjugate degrading map $\varrho_{\tilde{B}} \mapsto \varrho_{E'}$. ∎

Conjecture. *Universal quantum cloning machines for qubits are conjugate degradable.*

The appendix contains some arguments supporting the conjecture.

To calculate the quantum capacity of $\mathcal{C}l_{N,M}$ in these cases it therefore suffices to maximize the single-letter coherent information of formula (4). In the case of the $1 \to 2$ UQCM, a convexity argument showed that optimum was achieved for the maximally mixed input. A similar argument applies here. The only difference is that instead of transforming according to the defining representation of $SU(2)$, the input A now transforms according to a different irreducible representation r of $SU(2)$. Let $|\psi\rangle \in A$ and $V : B \to E'F$ be the Stinespring dilation of the conjugate degrading map. As before, let $\omega(\psi) = VU\psi U^\dagger V^\dagger$. For a qubit state $|\phi\rangle$, unitary covariance gives

$$\begin{aligned} H(F|E')_{\omega(\phi^{\otimes N})} &= \int_{SU(2)} H(F|E')_{\omega(r(W)\phi^{\otimes N} r(W)^\dagger)} dW \\ &\le H(F|E')_{\omega(\int r(W)\phi^{\otimes N} r(W)^\dagger dW)} \\ &= H(F|E')_{\omega(\Pi^N/(N+1))}, \end{aligned}$$

where Π^N is the projector onto the symmetric subspace of N qubits. The unitary covariance of the channel and the fact that the output representation is also irreducible implies that $\mathcal{C}l_{N,M}(\Pi^N/(N+1))$ is itself maximally mixed and therefore equal to $\Pi^M/(M+1)$. It therefore suffices to determine the output $\xi = \mathcal{N}^c(\Pi^N/(N+1))$ of the complementary channel.

The Stinespring dilation of the Gisin-Massar cloning machine acts as follows [14]:

$$U_{NM} : |N-k:0,k:1\rangle_A \mapsto \sum_{j=0}^{M-N} \sqrt{\alpha_{kj}}\,|M-k-j:0,k+j:1\rangle_B\,|P_j\rangle_E\,, \tag{21}$$

where both $|M-k-j:0,k+j:1\rangle_B$ and $|P_j\rangle_E$ form orthogonal sets and

$$\sqrt{\alpha_{kj}} = \sqrt{\frac{(M-N)!(N+1)!}{k!(N-k)!(M+1)!}}\sqrt{\frac{(k+j)!(M-k-j)!}{j!(M-N-j)!}}. \tag{22}$$

Therefore, an input maximally mixed state transforms as

$$\begin{aligned} &\bigoplus_{k=0}^{N} |N-k:0,k:1\rangle\langle N-k:0,k:1|_A \mapsto \\ &\qquad \bigoplus_{k=0}^{N}\sum_{j=0}^{M-N} \alpha_{kj}|M-k-j:0,k+j:1\rangle\langle M-k-j:0,k+j:1|_B \otimes |P_j\rangle\langle P_j|_E. \end{aligned} \tag{23}$$

Hence tracing over the B subsystem (for every k) results in a diagonal matrix. Evaluating the partial trace from Eq. (21) reveals that the $j-$th diagonal element of the output density matrix is of the form $\xi_{jj} = \sum_{k=0}^{N} \alpha_{kj}^2$. Using the identity

$$\alpha_{kj} = \frac{N+1}{M+1}\frac{\binom{N}{k}\binom{M-N}{j}}{\binom{M}{k+j}} = \frac{N+1}{M+1}\frac{1}{\binom{M}{N}}\binom{k+j}{k}\binom{M-k-j}{N-k} \tag{24}$$

gives a relatively compact expression for the diagonal entries of ξ:

$$\xi_{jj} = \frac{N+1}{M+1} \frac{1}{\binom{M}{N}} \sum_{k=0}^{N} \binom{k+j}{k} \binom{M-k-j}{N-k}. \tag{25}$$

The sum is reminiscent of Vandermonde's identity

$$\sum_{k=0}^{K} \binom{N}{k} \binom{M-N}{K-k} = \binom{M}{K}$$

but more complicated because of the presence of k in the 'numerator'. It can nonetheless be evaluated using generating functions. Recall the power series [16, p. 336]

$$\frac{1}{(1-z)^{1+j}} = \sum_{k=0}^{\infty} \binom{k+j}{j} z^k = \sum_{k=0}^{\infty} \binom{k+j}{k} z^k. \tag{26}$$

Letting $l = M - N - j$ we get

$$\xi_{jj} \propto \sum_{k=0}^{N} \binom{k+j}{k} \binom{N-k+l}{N-k}. \tag{27}$$

But then we have

$$\frac{1}{(1-z)^{1+j}} \frac{1}{(1-z)^{1+l}} = \frac{1}{(1-z)^{2-N-M}} \tag{28}$$

which implies via the power series expansion that ξ_{jj} is independent of j so the diagonal matrix is a multiple of identity. Examining the coefficients in more detail yields

$$\sum_{k=0}^{N} \binom{k+j}{k} \binom{M-k-j}{N-k} = \binom{1+M}{N}$$

and thus

$$\xi_{jj} = \frac{N+1}{M-N+1}. \tag{29}$$

Having demonstrated that both the output and environment density operators are both maximally mixed when the input is, we get our final result:

$$Q^{(1)}(\mathcal{C}l_{N,M}) = \log(M+1) - \log(M-N+1) = \log \frac{M+1}{M-N+1}. \tag{30}$$

This formula generalizes the one found for the $1 \to 2$ UQCM in Eq. (10) and, by virtue of their conjugate degradability, gives the exact quantum capacity for $N \to N+1$ and $1 \to M$ cloners. We see that the quantum capacity is strictly positive for all $M \geq N$: all the optimal quantum cloning machines are capable of transmitting quantum information reliably. Of course, as M/N increases, the maximized coherent information decreases, going to zero as M/N goes to infinity. This formula complements the calculation of the classical capacity for all $1 \to M$ cloning machines presented in [3].

IV. GENERAL PROPERTIES OF CONJUGATE DEGRADABLE CHANNELS

A recent paper by Cubitt *et al.* provided a detailed investigation of the properties of degradable channels [9]. In this section, we reproduce a number of their results, demonstrating that conjugate degrable channels have many of the same properties. Most of the proofs are very similar to those in the original paper with only minor modifications required to accommodate the complex conjugation.

Throughout the section, we will abbreviate $\dim(X)$ to d_X.

Theorem 2 (Compare to Theorem 1 of [9]). *Let $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$ be a channel mapping every pure state to pure state. Then, either*

(i) *$d_A \leq d_B$ and $\mathcal{N}(\varrho) = U\varrho U^\dagger$, with partial isometry U such that $U^\dagger U = I_{d_A}$, is always conjugate degradable with Choi rank $d_E = 1$, or*

(ii) *$\mathcal{N}(\varrho) = (\mathrm{Tr}\,\varrho)|\phi\rangle\langle\phi|$ for all ϱ. In this case, the channel $\mathcal{N}$ is conjugate antidegradable and maps every state to a single fixed pure state.*

Proof. The channel $\mathcal{N}$ can be expressed in its Stinespring dilation form by $\mathcal{N}(\varrho) = \mathrm{Tr}_E\, U\varrho U^\dagger$, with $U : A \to BE$ satisfying $U^\dagger U = I$. Let $\{|\alpha_k\rangle\}$ be an orthonormal basis of A. Since the channel maps every pure state to a pure state, we have $U\,|\alpha_k\rangle = |\beta_k\rangle \otimes |\gamma_k\rangle$ for some states $|\beta_k\rangle$ and $|\gamma_k\rangle$. Isometries preserve inner products, and so we must have $\langle\beta_j|\beta_k\rangle\langle\gamma_j|\gamma_k\rangle = \delta_{jk}$. For any $j \neq 1$, define the state $\phi_1 \equiv \frac{1}{2}(|\alpha_1\rangle + |\alpha_j\rangle)(\langle\alpha_1| + \langle\alpha_j|)$. If we send ϕ_1 through the channel $\mathcal{N}$, we get an output state of the form

$$\mathcal{N}(\phi_1) = \frac{1}{2}(|\beta_1\rangle\langle\beta_1| + \langle\gamma_1|\gamma_j\rangle|\beta_1\rangle\langle\beta_j| + \langle\gamma_j|\gamma_1\rangle|\beta_j\rangle\langle\beta_1| + |\beta_j\rangle\langle\beta_j|) \tag{31}$$

If $|\gamma_1\rangle$ and $|\gamma_j\rangle$ are orthogonal, the output state is $\mathcal{N}(\phi_1) = \frac{1}{2}|\beta_1\rangle\langle\beta_1| + \frac{1}{2}|\beta_j\rangle\langle\beta_j|$, which is pure if and only if $|\beta_j\rangle = |\beta_1\rangle$. Therefore, we must have $|\beta_j\rangle = |\beta_1\rangle$ whenever $\langle\gamma_1|\gamma_j\rangle = 0$.

Suppose the latter does not hold. Then we have $\langle\beta_1|\beta_j\rangle = 0$, and the output state is pure if and only if $|\langle\gamma_1|\gamma_j\rangle| = 1$, or, equivalently, $|\gamma_j\rangle = e^{i\theta}\,|\gamma_1\rangle$. Thus, when no states γ_j are orthogonal to γ_1, the channel $\mathcal{N}$ is of the form (i) (i.e $d_A \leq d_B$ and $d_E = 1$). The complementary channel $\mathcal{N}^c$ is given by $\mathcal{N}^c(\varrho) = (\mathrm{Tr}\,\varrho)|\gamma_1\rangle\langle\gamma_1|$. Since $\mathrm{Tr}(U\varrho U^\dagger) = \mathrm{Tr}\,\varrho$ for all $\varrho \in A$, the channel $\mathcal{N}$ is conjugate degradable with conjugate degrading map $\mathcal{D}(\sigma) = (\mathrm{Tr}\,\sigma)|\gamma_1\rangle\langle\gamma_1|^T$.

Suppose now instead that $d_E \neq 1$. From the previous paragraph, we may assume without loss of generality that $\langle\gamma_1|\gamma_2\rangle = 0$. We prepare a state $\phi_2 = \frac{1}{2}(|\alpha_2\rangle + |\alpha_j\rangle)(\langle\alpha_2| + \langle\alpha_j|)$, and send it through the channel $\mathcal{N}$. To obtain a pure output, we must have $|\gamma_j\rangle = e^{i\theta}\,|\gamma_2\rangle$ for any j such that $\langle\gamma_1|\gamma_j\rangle \neq 0$ (see Eq. (31)). But $|\gamma_j\rangle$ cannot be proportional to two orthogonal vectors, and so we must have $\langle\gamma_1|\gamma_j\rangle = 0$ for all j. The only possible CPTP map is then $\mathcal{N}(\varrho) = (\mathrm{Tr}\,\varrho)|\beta_1\rangle\langle\beta_1|$. The complementary channel $\mathcal{N}^c$ is the identity channel. The channel $\mathcal{N}$ is conjugate antidegradable with conjugate degrading map $\mathcal{D}(\sigma) = (\mathrm{Tr}\,\sigma)|\beta_1\rangle\langle\beta_1|^T$. ■

Lemma 1 (Compare to Lemma 2 of [9]). *Let $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$ be conjugate degradable, and for a pure state $|\psi_j\rangle$ define $B_j = \mathrm{range}\ \mathcal{N}(|\psi_j\rangle\langle\psi_j|)$ and $E_j = \mathrm{range}\ \mathcal{N}^C(|\psi_j\rangle\langle\psi_j|)$. Then, the spaces B_j and E_j have equal dimensions: $d_{B_j} = d_{E_j}$. Furthermore, if the vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_m\rangle$ satisfy* span $\bigcup_j B_j = B$, *we also have* span $\bigcup_j E_j = E$ *provided d_E is the Choi rank of the channel.*

Proof. Let $U : A \to BE$ be the partial isometry in the Stinespring dilation of the channel $\mathcal{N}$. The first part of the lemma can be seen to hold by looking at the Schmidt decomposition of $U\,|\psi_j\rangle$:

$$U\,|\psi_j\rangle = \sum_{k=1}^{r_j} \mu_{jk}\,|\phi_k^j\rangle \otimes |\omega_k^j\rangle\,, \tag{32}$$

where for each j $\{|\phi_k^j\rangle\}$ and $\{|\omega_k^j\rangle\}$ are sets of orthonormal states of the B and E spaces respectively. Since the states $\{|\phi_k^j\rangle\}$ (resp. $\{|\omega_k^j\rangle\}$) are eigenvectors of $\mathcal{N}(|\psi_j\rangle\langle\psi_j|)$ (resp. $\mathcal{N}^c(|\psi_j\rangle\langle\psi_j|)$), we have $r_j = d_{B_j} = d_{E_j}$.

We prove the second part of the lemma by contradiction. Assume span $\bigcup_j E_j \neq E$. Then there exists a state $|\omega^\perp\rangle \in E$ orthogonal to span$\{|\omega_k^j\rangle : j = 1\ldots m, k = 1\ldots r_j\}$. Since $\mathcal{N}$ is conjugate degradable, we have

$$\mathcal{N}^C(|\psi_j\rangle\langle\psi_j|) = \sum_{k=1}^{r_j} \mu_{jk}^2 |\omega_k^j\rangle\langle\omega_k^j| = \mathcal{C}\circ\mathcal{D}\circ\mathcal{N}(|\psi_j\rangle\langle\psi_j|) = \sum_{k=1}^{r_j} \mu_{jk}^2 (\gamma_k^j)^T, \tag{33}$$

where $\gamma_k^j \equiv \mathrm{Tr}_G\, V|\phi_k^j\rangle\langle\phi_k^j|V^\dagger$ and the operator $V : B \to EG$ is the partial isometry in the Stinespring dilation of the conjugate degrading channel $\mathcal{D}$.

From Eq. (33) and the fact that γ_k^j is positive semidefinite, we have $0 = \mathrm{Tr}[(\gamma_k^j)^T|\omega^\perp\rangle\langle\omega^\perp|] = \mathrm{Tr}[\gamma_k^j|\omega^\perp\rangle\langle\omega^\perp|^T]$ for all j,k. Consequently, $(|\omega^\perp\rangle\langle\omega^\perp|^T \otimes I_G)V\,|\phi_k^j\rangle = 0$ for all j,k. By hypothesis, however, any $|\phi\rangle \in B$ can be written as a superposition of the $|\phi_k^j\rangle$. It follows that $\langle\omega^\perp|\,\mathcal{D}(|\phi\rangle\langle\phi|)^T\,|\omega^\perp\rangle = 0$ for any $|\phi\rangle \in B$. This implies that $\langle\omega^\perp|\,\mathcal{N}^c(\psi)\,|\omega^\perp\rangle = 0$ for all states ψ, contradicting the fact that d_E is the Choi rank of $\mathcal{N}$. Hence, span $\bigcup_j E_j = E$. ■

Theorem 3 (Compare to Theorem 3 of [9])**.** *Let $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$ be a CPTP map which sends at least one pure state $|\psi\rangle$ to an output state $\varrho = \mathcal{N}(|\psi\rangle\langle\psi|)$ with full rank, i.e,* rank$\mathcal{N}(\varrho) = d_B$*. If $\mathcal{N}$ is conjugate degradable, we have $d_B = d_E$.*

Proof. The hypothesis of Lemma 1 is satisfied with $m = 1$ so that $E =$ range $\mathcal{N}^c(|\psi\rangle\langle\psi|)$. Then, using the first part of Lemma 1, we get

$$d_B = \dim[\text{range } \mathcal{N}(|\psi\rangle\langle\psi|)] = d_{B_1} = d_{E_1} = \dim[\text{range } \mathcal{N}^c(|\psi\rangle\langle\psi|)] = d_E.$$

■

Theorem 4 (Compare to Theorem 4 of [9])**.** *Let $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$ be a conjugate degradable channel with qubit output. Then, the following two properties always hold:*

(i) *The Choi rank of the channel $\mathcal{N}$ is at most two, and*

(ii) *The dimension of the input system $\mathcal{H}_A$ is at most three.*

Proof. For the first part, assume that the Choi rank is greater than two. Then, by Theorem 3, every pure state must be mapped to a rank 1 output. However, Theorem 2 combined with the fact that $\mathcal{N}$ is conjugate degradable then implies that the Choi rank must be one. The second part of the proof again follow the same lines as the proof of Theorem 4 [9] but we omit it for brevity. ■

Lemma 2 (Compare to Lemma 17 of [9])**.** *Let $\mathcal{N} : \mathcal{B}(A) \to \mathcal{B}(B)$ be a conjugate antidegradable channel and $\Delta : \mathcal{B}(B) \to \mathcal{B}(C)$ any CPTP map. Then the channel $\Delta\circ\mathcal{N}$ is also conjugate antidegradable.*

Proof. The proof follows the same lines as in Lemma 17 of [9]. The key difference is the addition of the operator $\mathcal{C}$ into Eq. (A7) of [9]. Since $\mathcal{N}$ is conjugate degradable, we replace Eq. (A7) with

$$(\Delta\circ\mathcal{C}\circ\mathcal{D}^a\circ\mathrm{Tr}_D)(\Delta\circ\mathcal{N})^c(\varrho) = \Delta\circ\mathcal{N}(\varrho), \tag{34}$$

where $\mathcal{D}^a$ is a conjugate degrading map for the complementary channel $\mathcal{N}^c$ and the partial trace operator Tr_D is taken over the environment D of the channel Δ. Using the Kraus representation of Δ and the fact that $\mathcal{C}(\varrho) = \varrho^T$ for any density operator ϱ, we have

$$\begin{aligned}(\Delta \circ \mathcal{C})(\varrho) &= \sum_k E_k \varrho^T E_k^\dagger \\ &= \left[\sum_k (E_k^\dagger)^T \varrho E_k^T\right]^T \\ &= (\mathcal{C} \circ \Delta')(\varrho)\end{aligned} \tag{35}$$

where Δ' is a map with Kraus elements $\{(E_k^\dagger)^T\}$. We have

$$\sum_k (E_k^\dagger)^T E_k^T = \sum_k (E_k E_k^\dagger)^T = I \tag{36}$$

and, thus, Δ' is also a CPTP map. Combining this result with Eq. (34) proves that $\Delta \circ \mathcal{N}$ is conjugate antidegradable. ■

Theorem 5 (Compare to Theorem 18 of [9])**.** *The set of conjugate antidegradable channels is convex.*

Proof. Follows easily from the proof of Theorem 18 in [9] by replacing all antidegradable channels with conjugate antidegradable ones. ■

The last result of this section is comparable to Theorem 5 of [9], which is itself distilled from an article of Wolf and Perez-Garcia [27]. Unlike the original, however, which demonstrated that qubit channels of Choi rank 2 are necessarily either degradable or antidegradable, it is possible for such channels to be neither conjugate degradable nor conjugate antidegradable.

Theorem 6 (Compare to Theorem 5 of [9])**.** *Let $\mathcal{N}$ be a qubit channel of Choi rank two. Then*

(i) *$\mathcal{N}$ is entanglement breaking if and only if it is conjugate antidegradable, and*

(ii) *$\mathcal{N}$ is cannot be both conjugate degradable and conjugate antidegradable.*

Proof. First, we recall the observation of [21] that, up to unitary transformations of the input and output, every Choi rank two qubit channel has the form

$$\mathcal{N}(\varrho) = A_+ \varrho A_+^\dagger + A_- \varrho A_-^\dagger, \tag{37}$$

where

$$\begin{aligned} A_+ &= \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix} \\ A_- &= \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix}. \end{aligned} \tag{38}$$

(i) Reformulating in terms of the Jamiołkowski isomorphism, the channel $\mathcal{N}$ is entanglement breaking if and only if for a fixed unnormalized maximally entangled state $|\Phi\rangle = |00\rangle + |11\rangle$,

$$[(\mathcal{N} \otimes \mathrm{id})(\Phi)]^{T_1} \geq 0. \tag{39}$$

(T_1 is partial transposition over the first subsystem). This inequality is equivalent to $1 - \sin^2\alpha - \sin^2\beta = 0$, whose solutions are $\alpha = (\pi/2 + k\pi) \pm \beta$, where $k \in \mathbb{Z}$. We will show that an identical condition characterizes conjugate antidegradability.

To that end, suppose the channel $\mathcal{N}$ is conjugate antidegradable: on the span of $\cup_\psi \mathcal{N}^c(\psi)$,

$$\mathcal{C} \circ \mathcal{D}^a = \mathcal{N} \circ (\mathcal{N}^c)^{-1}, \tag{40}$$

where $\mathcal{D}^a$ is a conjugate antidegrading map. Following [27], this can be rewritten as an equation in the corresponding Jamiołkowski matrices:

$$R_{\mathcal{C}\circ\mathcal{D}^a} = \left(R^\Gamma_{\mathcal{N}} \left(R^\Gamma_{\mathcal{N}^c}\right)^{-1} \right)^\Gamma, \tag{41}$$

where $R_{\mathcal{M}} = [(\mathcal{M} \otimes \mathrm{id})(\Phi)]$ is the Jamiołkowski matrix of the map $\mathcal{M}$ and Γ is an involution operation [27] defined as $\langle ij|R^\Gamma_{\mathcal{N}}|jk\rangle = \langle ik|R_{\mathcal{N}}|jl\rangle$. Since $\mathcal{C}$ is density matrix transposition, $R_{\mathcal{C}\circ\mathcal{D}^a} = (R_{\mathcal{D}^a})^{T_1}$. It follows that

$$R_{\mathcal{D}^a} = \left[\left(R^\Gamma_{\mathcal{N}} \left(R^\Gamma_{\mathcal{N}^c}\right)^{-1} \right)^\Gamma \right]^{T_1}. \tag{42}$$

A calculation then reveals that the spectrum of $R_{\mathcal{A}}$ is

$$\lambda_{\mathcal{D}^a} \in \left\{ 1, 1, \pm \frac{\sin^2\beta + \sin^2\alpha - 1}{\sin^2\beta - \sin^2\alpha} \right\} \tag{43}$$

and we immediately see that the operator $\mathcal{D}^a$ represents a CP map if and only if the numerator of the last two eigenvalues is zero, which coincides precisely with the condition found in Eq. (39) for the positivity of the partial transpose of the output of $\mathcal{N}$.

(ii) Suppose now there exists a degrading map $\mathcal{D}$ up to complex conjugation $\mathcal{C}$

$$\mathcal{C} \circ \mathcal{D} = \mathcal{N}^c \circ \mathcal{N}^{-1}. \tag{44}$$

Following the method of the previous part, we find the eigenvalues

$$\lambda_{\mathcal{D}} \in \left\{ 1, 1, \pm \frac{\sin^2\beta - \sin^2\alpha}{\sin^2\beta + \sin^2\alpha - 1} \right\}. \tag{45}$$

We see that when $\mathcal{D}$ is a CP map the eigenvalues of $\mathcal{D}^a$ blow up, indicating that the map $\mathcal{D}^a$ cannot be defined. Likewise, when $\mathcal{D}^a$ is a CP map, the eigenvalues of $\mathcal{D}$ blow up. ■

V. DISCUSSION

We have introduced the property of conjugate degradability and demonstrated that conjugate degradable quantum channels enjoy a single-letter quantum capacity formula. We then identified some natural examples of conjugate degradable channels: an Unruh channel as well as both $N \to N+1$ and $1 \to M$ universal quantum cloning machines. For each of these examples, we then evaluated the formulas to get simple expressions for the channels' quantum capacities. For the Unruh channel, the quantum capacity was strictly positive for all accelerations and for the cloning channels, it was strictly positive for all $M \geq N$.

We also verified that many of the properties of degradable channels discovered in [9] also hold for conjugate degradable channels. For example, the set of conjugate antidegradable channels is convex and closed under composition with arbitrary quantum channels. In fact, it is consistent

with our findings that the conjugate degradable channels form a subset of the degradable channels. Determining whether that is the case is the most important open problem arising from this work. Any channels that are conjugate degradable but not degradable would necessarily have entanglement-binding complementary channels and therefore be at least moderately exotic.

We hope that conjugate degradability will prove to be a useful concept regardless of the resolution of its relationship to degradability. The conjugate degradability of the example channels considered in this article is essentially obvious. On the other hand, the $1 \to M$ universal quantum cloning machines are also known to be degradable for $M \leq 6$, but demonstrating that fact requires a tedious calculation and it isn't clear how to generalize the calculation to general cloning machines [3]. More conjugate degradable channels are certainly waiting to be discovered. We encourage readers to find them.

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Appendix

This appendix provides some motivation for the conjecture following Theorem 1. It also contains some details of the calculations used in our study of cloning channels.

The action of the Stinespring dilation for the universal qubit cloning machine ($\mathcal{C}l$) on N identical input qubits $|N:\psi\rangle$ can be written as [15]

$$\begin{aligned} |N:\psi\rangle_A \xrightarrow{U_{N,M}} & \sum_{j=0}^{M-N} \sqrt{\alpha_j(N,M)}\, |M-j:\psi, j:\psi^\perp\rangle_B\, |M-j-1:\bar{\psi}, j:\bar{\psi}^\perp\rangle_E \\ & \equiv \sum_{j=0}^{M-N} \sqrt{\alpha_j(N,M)}\, |S_j\rangle_B\, |\bar{R}_j\rangle_E\,, \end{aligned} \quad (46)$$

where

$$\sqrt{\alpha_j(N,M)} = \sqrt{\frac{N+1}{M+1}}\sqrt{\frac{(M-N)!(M-j)!}{(M-N-j)!M!}}. \quad (47)$$

We write $|N-j:\psi, j:\psi^\perp\rangle$ for the normalized completely symmetric state (CSS) of N qubits with exactly $N-j$ of them occupying state ψ.

First, we can transform the complementary channel output by complex conjugation $E \mapsto E'$ simply by removing the bar from $|\bar{R}_j\rangle_E$. That is, we replace $|\bar{R}_j\rangle = |M-j-1:\bar{\psi}, j:\bar{\psi}^\perp\rangle_E$ by $|R_j\rangle = |M-j-1:\psi, j:\psi^\perp\rangle_E$. Inspecting the density matrices of the B and E' subsystems then reveals a kind of Schmidt form (heretofore dropping the explicit dependence of the diagonal coefficients on N and M):

$$\varrho_B = \bigoplus_{j=0}^{M-N} \alpha_j |S_j\rangle\langle S_j|, \qquad \varrho_{E'} = \bigoplus_{j=0}^{M-N} \alpha_j |R_j\rangle\langle R_j|. \quad (48)$$

Now trace over a qubit in the B system, for comparison with the state on E':

$$\varrho_B \to \varrho_{\tilde{B}} = \bigoplus_{j=0}^{M-N} \beta_j |R_j\rangle\langle R_j|. \tag{49}$$

Some dull but straightforward algebra yields

$$\beta_j = \alpha_j \frac{M-j}{M} + \alpha_{j+1} \frac{1+j}{M} \tag{50}$$

if we set $\alpha_{M-N+1} = 0$.

We now show that, for all N to M UQCM, $\varrho_{\tilde{B}}$ majorizes $\varrho_{E'}$ in the sense that $\vec{\beta} \succ \vec{\alpha}$. (Let $\vec{\beta}^{\downarrow}$ be the vector with the same coefficients as $\vec{\beta}$ but arranged in nonincreasing order and likewise for $\vec{\alpha}^{\downarrow}$. Then $\vec{\beta} \succ \vec{\alpha}$ if and only if for all k, $\sum_{j=1}^{k} \beta_j^{\downarrow} \geq \sum_{j=1}^{k} \alpha_j^{\downarrow}$.) This majorization relation implies that for every $\varrho_{\tilde{B}}$ there exists a unital CPTP map transforming it into $\varrho_{E'}$ [1]. What we cannot conclude from this analysis, however, is that there is a single unital CPTP map transforming every $\varrho_{\tilde{B}}$ into $\varrho_{E'}$ as was the case with the depolarizing channel for the cloners from Theorem 1. Such a channel, if it exists, would be the necessary conjugate degrading map.

Lemma 3. *The coefficients α_j and β_j are in decreasing order, i.e $\alpha_j > \alpha_{j+1}$ and $\beta_j > \beta_{j+1}$.*

Proof. For α_j it can be seen directly from Eq. (47) that

$$\frac{\alpha_j}{\alpha_{j+1}} = \frac{M-j}{M-N-j}. \tag{51}$$

For β_j we first strip Eq. (50) of all factors independent of j and find

$$\beta_j \propto \frac{(M-N-j)!}{(M-N-j)!} \big((M-j)^2 + (M-N-j)(1+j) \big).$$

Then

$$\frac{\beta_j}{\beta_{j+1}} \propto \frac{M-j-1}{M-j-N} \frac{(M-j)^2 + (M-N-j)(1+j)}{(M-j-1)^2 + (M-N-j-1)j}$$

and so $\frac{\beta_j}{\beta_{j+1}} > 1$ if $\frac{M-j-1}{M-N-j} \geq 1$ which is always true. ■

Remark. *This result is physically sensible: states similar to the input state occur with higher probability. These are the states in which fewer qubits are rotated with respect to the input state.*

Lemma 4. $\vec{\beta} \succ \vec{\alpha}$

Proof. Since the coefficients of $\vec{\alpha}$ and $\vec{\beta}$ are already ordered, it suffices to calculate:

$$\begin{aligned}
\sum_{j=0}^{k} \beta_j &= \sum_{j=0}^{k} \left[\alpha_j \frac{M-j}{M} + \alpha_{j+1} \frac{1+j}{M} \right] \\
&= \alpha_0 + \sum_{j=1}^{k} \alpha_j \left[\frac{M-j}{M} + \frac{j}{M} \right] + \alpha_{k+1} \frac{k+1}{M} \\
&= \sum_{j=0}^{k} \alpha_j + \alpha_{k+1} \frac{k+1}{M}.
\end{aligned}$$

Since $\alpha_{k=1} \geq 0$, we get $\sum_{j=0}^{k} \beta_j \geq \sum_{j=0}^{k} \alpha_j$, as required. ■

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